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FREE VIBRATION ANALYSIS OF BEAMS ON A PASTERNAK FOUNDATION USING LEGENDRE POLYNOMIALS AND RAYLEIGHT-RITZ METHOD

А. Рахбар-Ранжі, А. Шахбазтабар. **Аналіз вільних вібрацій балок на основі Пастернака з використанням полиномов Лежандра і метода Релея-Рітца.** Досліджується вільна вібрація балок Ейлера-Бернуллі і Тимошенко, що спираються на двопараметричну пружню основу Пастернака. Для виведення керуючого рівняння використовується метод Релея-Рітца, а многочлени Лежандра, помножені на граничну функцію, використовується в якості допустимих функцій для визначення полів зсуву. Точність результатів оцінюється в порівнянні з даними, доступними в літературі. Показано, що метод має хорошу конвергенцію незалежно від теорії балок, граничних умов і параметрів пружної основи. Наведені природні частоти вібрації балок з різними граничними умовами, параметри пружної основи і відносини висоти до довжини.
*Ключові слова:*вільна вібрація, метод Релея-Рітца, теорія балок Ейлера-Бернуллі, теорія балок Тимошенко, пружня основа Пастернака

A. Rahbar-Ranji, A. Shahbaztabar. **Free vibration analysis of beams on a Pasternak foundation using Legendre polynomials and Rayleigh-Ritz method.** A free vibration of Euler-Bernoulli and Timoshenko beams resting on a two-parameter elastic foundation of Pasternak type has been investigated. Rayleigh-Ritz method is employed to deduce the governing equation and the Legendre polynomials multiplied by a boundary function is used as admissible functions to define the displacement fields. Accuracy of the results is evaluated by comparing with those available in the literature. It is shown that the method has a good and rapid convergence regardless of the beam theory, boundary conditions and elastic foundation parameters. Natural frequencies of beams with different boundary conditions, elastic foundation parameters, and ratios of height-to-length are presented.
Keywords: Free vibration; Rayleigh-Ritz method; Euler-Bernoulli's beam theory; Timoshenko beam theory; Pasternak elastic foundation

Introduction. Beams and plates resting on elastic foundations have many applications in all branches of engineering. Building foundations, railroad ties, and engine foundations are some examples of beam/plate resting on elastic foundations. The simplest model of elastic foundation was introduced by Winkler [1]. In this model a vertical displacement of the foundation is assumed to be proportional to the contact pressure and the proportionality constant is called the modulus of foundation. In the Winkler type of elastic foundation there is no mutual interaction between adjacent springs. Different models have been introduced to consider this interaction. Pasternak [2] is a two-parameter elastic foundation that interaction is accomplished by connecting the ends of the springs to a shear layer.

Depending upon the ratio of height-to-length of beams, different theories are used to express the kinematic of deformation. For beams with a low ratio of height-to-length, Euler-Bernoulli Beam Theory (EBBT), in which shear deformation is ignored and planes normal to the longitudinal fibers remain plane and normal after deformation, should be used. For beams with a high ratio of height-to-length, Timoshenko Beam Theory (TBT), in which plane cross sections remain plane but not perpendicular to longitudinal, is used [3, 4].

Many investigations have been carried out to study free vibrations of both EBBs and TBs on an elastic foundation. Eisenberger and Clastornik [5] solved the eigenvalue problems of vibration and stability of an EBB resting on a variable elastic foundation. De Rosa [6] investigated the stability and dynamics of beams on a Winkler type elastic foundation by the cell discretization method. Zhou [7] presented a general solution to vibration of EBBs on the variable Winkler type of elastic foundations. Naidu and Rao [8] have studied the vibration of an uniform EBB initially stressed on Winkler and Pas-

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ternak types of foundation. Franciosi and Masi [9] have used the Finite Element Method (FEM) to study EBBs on elastic foundations of the variable Winkler type. Yokoyama and Suita [10] have obtained the natural frequencies and transient responses of TBs resting on a two-parameter elastic foundation using FEM. Yokoyama [11] has obtained the parametric instability of a TB resting on a Winkler type elastic foundation using FEM. Abbas and Thomas [12] have developed a FEM for stability analysis of TBs resting on an elastic foundation subjected to periodic axial loads.Thambiratnam and Zhuge [13] have used a simple FEM for free vibration analysis of EBBs on an elastic foundation. Balkaya et al. [14] have used the differential transform method to analyse the free vibration of EBBs/TBs resting on a elastic soil. Wang et al. [15] have presented an exact solution for TBs on various elastic foundations using Green's function.

In the present paper, a free vibration of EBBs/TBs resting on a Pasternak type elastic foundation has been investigated. Rayleigh-Ritz method is employed to derive the governing equation and the Legendre polynomials multiplied by a boundary function are used as admissible functions to study the free vibration. Convergence and applicability of the method are demonstrated through some numerical examples.

Theory and formulation. Let's assume a straight uniform beam of length L, depth H with a rectangular cross-section of unit width which has rested on an elastic foundation of Pasternak type (Fig. 1).

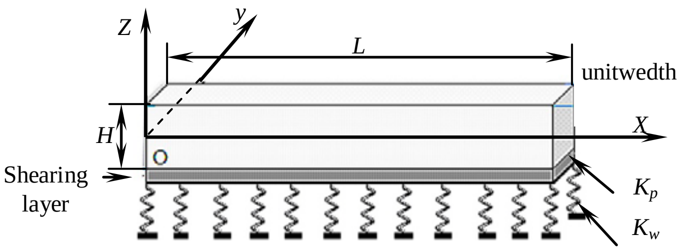

Fig. 1. A typical beam resting on an elastic foundation of Pasternak type

Displacement fields in x and z directions are defined as follows:

– EBBT:

$$u_x(x,z,t) = -z\frac{\partial w(x,t)}{\partial x}, \quad (1)$$

$$u_z(x,z,t) = w(x,t), \quad (2)$$

– TBT:

$$u_x(x,z,t) = -z\psi(x,t), \quad (3)$$

$$u_z(x,z,t) = w(x,t), \quad (4)$$

where $\psi(x,t)$ is the rotation function of the cross-section and $w(x,t)$ is transverse displacement function of the neutral axis. Stresses and strains are determined as follows:

– EBBT:

$$\varepsilon_x = -z\frac{\partial^2 w(x,t)}{dx^2}, \quad (5)$$

$$\sigma_x = -Ez\frac{\partial^2 w(x,t)}{dx^2}, \quad (6)$$

$$\gamma_{xz} = 0, \quad (7)$$

$$\tau_{xz} = 0, \quad (8)$$

– TBT:

$$\varepsilon_x = -z\frac{\partial\psi(x,t)}{dx}, \quad (9)$$

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$$\sigma_x = -Ez\frac{\partial\psi(x,t)}{dx}, \tag{10}$$

$$\gamma_{xz} = \frac{\partial w(x,t)}{dx} - \partial\psi(x,t), \tag{11}$$

$$\tau_{xz} = KG\gamma_{XZ}, \tag{12}$$

where ε_x is the normal strain, σ_x is the normal stress, γ_{xz} is the shear strain, τ_{xz} is the shear stress, E and G are the Young's modulus and shear modulus, respectively, and K is shear correction factor to account for non-uniform shear stress distribution.

The strain energy of a beam resting on Pasternak foundation is calculated as follows:

$$U = \frac{1}{2}\int_0^L\int_A(\sigma_x\varepsilon_x + \tau_{xz}\gamma_{xz})dAdx + \frac{k_w}{2}\int_0^L(w(x,t))^2dx + \frac{k_p}{2}\int_0^L\left(\frac{\partial w(x,t)}{\partial x}\right)^2dx, \tag{13}$$

where k_w and k_p are Winkler and shear layer elastic coefficient of foundation, respectively. Substituting Eqs. (5)–(12) into (13) and integrating with respect to *z*, yields to following expression for calculation of the strain energy:

– EBBT:

$$U^E = \frac{1}{2}\int_0^L EI\left(\frac{\partial w(x,t)}{\partial x}\right)^2dx + \frac{k_w}{2}\int_0^L\left(w(x,t)\right)^2dx + \frac{k_p}{2}\int_0^L\left(\frac{\partial w(x,t)}{\partial x}\right)^2dx, \tag{14}$$

– TBT:

$$U^E = \frac{1}{2}\int_0^L\left[EI\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2 + KAG\left(\frac{\partial\psi(x,t)}{\partial x} - \partial\psi(x,t)\right)^2\right]dx + \\ + \frac{k_w}{2}\int_0^L\left(w(x,t)\right)^2dx + \frac{k_p}{2}\int_0^L\left(\frac{\partial w(x,t)}{\partial x}\right)^2dx, \tag{15}$$

which I is the second moment of inertia and A is the cross-section area. The kinetic energy of the beam is calculated as follows:

– EBBT:

$$T^E = \frac{1}{2}\int_0^L\rho A\left(\frac{\partial w(x,t)}{\partial t}\right)^2dz, \tag{16}$$

– TBT:

$$T^T = \frac{1}{2}\left[\int_0^L\rho A\left(\frac{\partial w(x,t)}{\partial t}\right)^2 + \rho I\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2\right]dx, \tag{17}$$

where ρ is the mass density of the beam. Following displacement functions are assumed:

$$\psi(x,t) = \Psi(x)\cos\omega t, \tag{18}$$

$$w(x,t) = W(x)\cos\omega t, \tag{19}$$

where $\Psi(x)$ and $W(x)$ are mode shape functions and trigonometric term shows that the response of the free vibration is harmonic with natural frequency, ω. Substituting Eqs. (18–19) into (14–17) yield to following expressions for maximum strain and kinetic energies:

– EBBT:

$$U_{\max}^{E}=\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 W(x)}{\partial x^2}\right)^2 dx+\frac{k_w}{2}\int_0^L (W(x,t))^2\,dx+\frac{k_p}{2}\int_0^L\left(\frac{\partial W(x)}{\partial x}\right)^2 dx\,, \tag{20}$$

$$T_{\max}{}^{T}=\frac{1}{2}\int_0^L \rho A(W(x))^2\,dx\,, \tag{21}$$

– TBT:

$$\begin{aligned}U_{\max}{}^{T}&=\frac{1}{2}\int_0^L\left[EI\left(\frac{\partial\Psi(x,t)}{\partial x}\right)^2+KAG\left(\frac{\partial W(x,t)}{\partial x}-\partial\Psi(x,t)\right)^2\right]dx+\\&+\frac{k_w}{2}\int_0^L (W(x,t))^2\,dx+\frac{k_p}{2}\int_0^L\left(\frac{\partial W(x,t)}{\partial x}\right)^2 dx,\end{aligned} \tag{22}$$

$$T_{\max}{}^{T}=\frac{\rho}{2}\int_0^L I\omega^2(\Psi(x))^2+A\omega^2(W(x))^2\,dx\,. \tag{23}$$

For convenience, the following non-dimensional parameters are introduced:

$$\chi=\frac{x}{L}\,, \tag{24}$$

$$\overline{W}=\frac{W}{L}\,, \tag{25}$$

$$\overline{\Psi}=\Psi\,, \tag{26}$$

The maximum total energy functional of the beam is defined as follows:

$$\Pi=U_{\max}-T_{\max}\,, \tag{27}$$

Substituting Eqs. (24–26) into (20–23), total energy functional of the beam is obtained as follows:

– EBBT:

$$\Pi^{E}=\frac{1}{2}\int_0^L\left(\frac{\partial^2\overline{W}}{\partial\chi^2}\right)d\chi-\frac{1}{2}\lambda^4\int_0^L(\overline{W}(\chi))^2 d\chi+\frac{\overline{K}_w}{2}\int_0^L(\overline{W}(\chi))^2 d\chi+\frac{\overline{K}_p}{2}\int_0^L\left(\frac{\partial\overline{W}(\chi)}{\partial\chi}\right)^2 d\chi\,, \tag{28}$$

– TBT:

$$\begin{aligned}\Pi^{T}&=\frac{1}{2}\int_0^L\left[\left(\frac{\partial\overline{\Psi}(\chi)}{\partial\chi}\right)^2+\overline{K}\left(\frac{\partial\overline{W}(\chi)}{\partial\chi}-\overline{\Psi}(\chi)\right)^2\right]d\chi-\\&-\frac{1}{2}\lambda^4\int_0^L\left[\Gamma\left(\frac{\partial\overline{\Psi}(\chi)}{\partial\chi}\right)^2+(\overline{W}(\chi))^2\right]d\chi+\frac{\overline{K}_w}{2}\int_0^L(\overline{W}(\chi))^2 d\chi+\frac{\overline{K}_p}{2}\int_0^L\left(\frac{\partial\overline{W}(\chi)}{\partial\chi}\right)^2 d\chi,\end{aligned} \tag{29}$$

where:

$$\overline{K}=\frac{KGAL^2}{EI}\,, \tag{30}$$

$$\lambda^4=\frac{\rho A\omega^2 L^2}{EI}\,, \tag{31}$$

$$\Gamma=\frac{1}{2}\left(\frac{H}{L}\right)^2, \tag{32}$$

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$$\overline{K}_w = \frac{k_w L^2}{EI}, \quad (33)$$

$$\overline{K}_p = \frac{k_w L^2}{EI\pi^2}, \quad (34)$$

where λ is non-dimensional natural frequency, $\overline{K}_w$ and $\overline{K}_p$ are non-dimensional Winkler and shearing layer elastic coefficient of foundation, respectively. Following relation is held:

$$\overline{K} = \frac{6K}{(1+\nu)}\left(\frac{L}{H}\right)^2. \quad (35)$$

Mode shape functions are assumed as follows:

– EBBT:

$$\overline{W}(\chi)^E = f_k \sum_{l=1}^{N} A_l g_l(2\chi-1), \quad (36)$$

– TBT:

$$\overline{\Psi}(\chi)^E = f_u \sum_{i=1}^{N} B_l g_l(2\chi-1), \quad (37)$$

$$\overline{W}(\chi)^T = f_v \sum_{j=1}^{N} C_l g_l(2\chi-1), \quad (38)$$

where A_l, B_i and C_j are the unknown constant coefficients, $g_m(2\chi-1)$ $(m=1,2,3,...)$ is the one-dimensional Legendre polynomial, N is the number of Legendre polynomial terms, and $f_s = \chi^r(\chi-1)^q$ $(s=k,u,v;\ r,q=0,1,2)$ are the boundary functions which at least should satisfy the essential boundary conditions. Table 1 shows the selected parameters for simply supported (SS) and clamped (CC) boundary conditions. The Legendre polynomial series is a set of orthogonal series in the interval 0…1 which yields rapid convergence and more accuracy. Fig. 2 shows the first six terms of this polynomial.

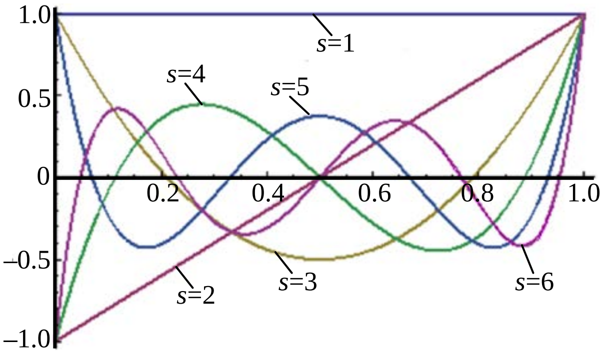

Fig. 2. The first six terms of the Legendre polynomials

Table 1

Boundary function elements for different boundary conditions

Boundary condition	f_k		f_u		f_v	
	r	q	r	q	r	q
SS	1	1	0	0	1	1
CC	2	2	1	1	1	1

The governing equation for free vibration analysis of EBB/TB is obtained by substituting Eqs. (36)–(38) into (28)–(29) and minimizing the total energy functional with respect to unknown coefficients as follows:

$$\frac{\partial \Pi}{dA_l}=0 \quad \frac{\partial \Pi}{dB_i}=0 \quad \frac{\partial \Pi}{dC_j}=0 \quad l,i,\ j=1,2,3,...,N. \quad (39)$$

The generalized form of eigenvalue problem would be as follows:

$$([k]-\lambda^4[m])\{\Delta\}=0, \quad (40)$$

where k and m are the stiffness and inertia matrices, respectively, Δ is the vector of unknown coefficients and $\lambda^2 = \omega L^2\sqrt{\dfrac{\rho A}{EI}}$.

Numerical and discussions. To demonstrate the convergence and applicability of the present method, free vibrations of EBB/TB are analyzed for different foundation parameters and boundary conditions. Table 2 gives convergence studies of current method for EBB with CC boundary condition and $\overline{K}_w$, $\overline{K}_p$. The number of Legendre polynomial terms has been increased from 2 to 18. It can be seen that, for the first modes of vibration with two numbers of the Legendre polynomial terms, a very good result is achieved. For a higher mode of vibration, the number of the Legendre polynomial terms should be increased. For example, for fifth mode, N equal to 10 yields an accurate result.

Table 2

Convergence study of an EBB resting on a Pasternak foundation
(CC boundary condition, $\overline{K}_w$, $\overline{K}_p = 2.5$*)*

№	Frequencyparameters				
	λ_1	λ_2	λ_3	λ_4	λ_5
2	5.32010	8.4285	–	–	–
4	5.32009	8.3825	11.6778	15.1898	–
6	5.32003	8.3815	11.4368	14.5382	19.0306
8	5.32003	8.3815	11.4281	14.4985	17.7129
10	5.32003	8.3815	11.4280	14.4977	17.5908
12	5.32003	8.3815	11.4280	14.4977	17.5862
14	5.32003	8.3815	11.4280	14.4977	17.5861

Table 3 shows convergence studies of the current method for TB with SS boundary conditions and ν=0.3, K=5/6, H/L=0.2, $\overline{K}_w = 10$, $\overline{K}_p = 1$.In this case, an accurate result is achieved for the first mode of vibration, with N equal to four, and for the fifth mode of vibration – with N equal to 10. It can be concluded that the current method has a good and rapid convergence irrespective of boundary conditions, beam theory and foundation parameters. For higher modes of vibration, more terms of the Legendre polynomial are needed.

Table 3

Convergence study of TB resting on Pasternak foundation
(SS boundary condition, ν=0.3, K=5/6, H/L=0.2, $\overline{K}_w$, $\overline{K}_p$*)*

№	Frequency parameters				
	λ_1	λ_2	λ_3	λ_4	λ_5
2	3.80593	–	–	–	–
4	3.71567	6.54319	9.1318	11.8962	–
6	3.71543	6.13311	8.2684	10.9306	13.0323
8	3.71543	6.12573	8.2306	10.1212	11.8064
10	3.71543	6.12570	8.2301	10.0342	11.6098
12	3.71543	6.12570	8.2301	10.0313	11.5987
14	3.71543	6.12570	8.2301	10.0313	11.5985

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Table 4 shows the first five natural frequencies of EBBs with SS boundary condition for different foundation parameters. The obtained results have been checked against Zhou [7] for the case of Winkler type of elastic foundation. Very good agreements can be seen regardless of foundation parameters and frequency number.

Table 4

Non-dimensional frequency parameters for EBB with SS boundary condition[*]

Foundation parameters		Frequency parameters				
$\overline{K}_w$	$\overline{K}_p$	λ_1	λ_2	λ_3	λ_4	λ_5
1	0	3.1496	6.28426	9.4251	12.5665	15.7080
	0.5	3.4827	6.4718	9.5533	12.6635	15.7860
	1	3.7408	6.6445	9.6766	12.7584	15.8628
	2.5	4.3002	7.0947	10.0206	13.0310	16.0868
10	0	3.2193 (3.220)	6.2932 (6.293)	9.4277 (9.427)	12.5676 (12.568)	15.7086 (15.709)
	0.5	3.5347	6.4801	9.5560	12.6646	15.7865
	1	3.7830	6.6522	9.6791	12.7595	15.8634
	2.5	4.3282	7.1010	10.0229	13.0320	16.0873
100	0	3.7484	6.3816	9.4545	12.5790	15.7144
		(3.748)	(6.382)	(9.454)	(12.579)	(15.715)
	0.5	3.9608	6.5613	9.5816	12.6757	15.7923
	1	4.1437	6.7273	9.7038	12.7703	15.8690
	2.5	4.5824	7.1630	10.0451	13.0421	16.0927
1000	0	5.7556	7.1121	9.7102	12.6905	15.7721
		(5.755)	(7.112)	(9.710)	(12.690)	(15.773)
	0.5	5.8184	7.2438	9.8277	12.7847	15.8491
	1	5.8793	7.3686	9.9412	12.8770	15.9250
	2.5	6.0513	7.7095	10.2601	13.1424	16.1464
10000	0	10.0243	10.3687	11.5652	13.6716	16.3167
	0.5	10.0363	10.4122	11.6354	13.7472	16.3863
	1	10.0483	10.4550	11.7043	13.8216	16.4551
	2.5	10.0842	10.5806	11.9042	14.0378	16.6563

Table 5 shows the first five natural frequencies of EBBs with CC boundary condition for different foundation parameters. The obtained results are examined against Franciosi and Masi [9], and very good agreements can be seen regardless of foundation parameters and frequency number.

[*] The numbers in the parentheses are taken from Zhou (1993)

Table 5

Non-dimensional frequency parameters for an EBB with CC boundary conditions[*]

Foundation parameters		Frequency parameters				
$\overline{K}_w$	$\overline{K}_p$	λ_1	λ_2	λ_3	λ_4	λ_5
1	0	4.7324	7.8537	10.9958	14.1372	17.2788
		(4.73)	(7.85)	(11.0)	–	–
	0.5	4.8691	7.9683	11.0864	14.2116	17.3416
		(4.87)	(7.97)	(11.09)	–	–
	1	4.9946	8.0780	11.1748	14.2847	17.4037
		(4.99)	(8.08)	(11.17)	–	–
	2.5	5.3200	8.3815	11.4280	14.4977	17.5861
		(5.32)	(8.38)	(11.43)	–	–
10	0	4.7535	7.8584	10.9975	14.1380	17.2792
	0.5	4.8885	7.9728	11.0880	14.2124	17.3420
	1	5.0125	8.0822	11.1765	14.2855	17.4041
	2.5	5.3350	8.3853	11.4295	14.4984	17.5865
100	0	4.9504	7.9043	11.0143	14.1460	17.2836
		(4.95)	(7.90)	(11.01)	–	–
	0.5	5.0707	8.0168	11.1045	14.2202	17.3463
		(5.23)	(8.16)	(11.24)	–	–
	1	5.1823	8.1245	11.1925	14.2932	17.4084
		(5.54)	(8.39)	(11.43)	–	–
	2.5	5.4773	8.4232	11.4446	14.5058	17.5907
		(5.48)	(8.42)	(11.44)	–	–
1000	0	6.2239	8.3251	11.1790	14.2248	17.3270
	0.5	6.2857	8.4218	11.2653	14.2978	17.3893
	1	6.3455	8.5150	11.3497	14.3696	17.4509
	2.5	6.5136	8.7768	11.5918	14.5790	17.6319
10000	0	10.1228	10.8392	12.5260	14.9493	17.7442
		(10.12)	(10.84)	(12.53)	–	–
	0.5	10.1374	10.8835	12.5876	15.0122	17.8022
		(10.16)	(10.94)	(12.68)	–	–
	1	10.1518	10.9272	12.6483	15.0744	17.8597
		(10.21)	(11.04)	(12.81)	–	–
	2.5	10.1943	11.0546	12.8252	15.2564	18.0287
		(10.41)	(11.38)	(13.21)	–	–

Tables 6–9 depict the first five frequencies of TBs resting on a Pasternak elastic foundation with SS/CC boundary conditions. In these tables the number of Legendre polynomial terms is taken as 14 and different ratios of height-to-length are considered. It can be concluded that for height-to-length ratios less than 0.05, this ratio has no influence on the results. However, the height-to-length ratio has an influence on the result when it is high. Its influence depends on boundary conditions, the mode number and foundation parameters. For example, when $\overline{K}_w = 1$, for SS boundary conditions, the first natural frequency decreases by 1.5 percent when the height-to-length ratio increases from 0.002 to 0.2. However, for CC boundary conditions and $\overline{K}_p = 0$, it decreases about 8% and, for $\overline{K}_p = 2.5$, it decreases by 10 %. Figure 3 shows the effect of the height-to-length ratio on the natural frequencies for the case $\overline{K}_w = 100$, $\overline{K}_p = 0.5$ and SS boundary condition. It can be seen that the effect of the height-

[*] The numbers in the parentheses are taken from Franciosi and Masi (1993)

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to-length ratio is more prominent for a higher mode of vibration, and when the ratio of height-to-length is less than 0.05, EBBT yields very accurate results.

Table 6

Non-dimensional frequency parameters for a TB with SS boundary conditions,
($\overline{K}_w = 1$, $\nu = 0.3$, $K = 5/6$, and N=14)

Height-to-lengthratio	Foundation parameters	Frequency parameters				
	$\overline{K}_p$	λ_1	λ_2	λ_3	λ_4	λ_5
0.002	0	3.1496	6.2841	9.4248	12.5658	15.7067
	0.5	3.4826	6.4718	9.5530	12.6628	15.7847
	1	3.7408	6.6444	9.6763	12.7577	15.8615
	2.5	4.3001	7.0946	10.0204	13.0303	16.0855
0.05	0	3.1430	6.2324	9.2557	12.1815	14.9927
	0.5	3.4772	6.4231	9.3887	12.2850	15.0790
	1	3.7360	6.5982	9.5164	12.3861	15.1637
	2.5	4.2960	7.0535	9.8713	12.6752	15.4098
0.10	0	3.1238	6.0917	8.8409	11.3433	13.6132
	0.5	3.4614	6.2914	8.9882	11.4660	13.7232
	1	3.7218	6.4737	9.1286	11.5850	13.8306
	2.5	4.2841	6.9442	9.5149	11.9213	14.1384
0.20	0	3.0540	5.6722	7.8400	9.6573	11.2222
	0.5	3.4048	5.9072	8.0403	9.8487	11.4145
	1	3.6720	6.1165	8.2264	10.0292	11.5972
	2.5	4.2427	6.6408	8.7186	10.5174	12.0967

Table 7

Non-dimensional frequency parameters for a TB with SS boundary conditions,
($\overline{K}_w = 10$, $\nu = 0.3$, $K = 5/6$, and N=14)

Height-to-lengthratio	Foundation parameters	Frequency parameters				
	$\overline{K}_p$	λ_1	λ_2	λ_3	λ_4	λ_5
0.002	0	3.2193	6.29315	9.4275	1.5670	15.7073
	0.5	3.5347	6.4801	9.5556	12.6640	15.7852
	1	3.7830	6.6521	9.6788	12.7588	15.8620
	2.5	4.3282	7.1010	10.0226	13.0313	16.0861
0.05	0	3.2130	6.2416	9.2585	12.1827	14.9933
	0.5	3.5294	6.4315	9.3914	12.2862	15.0796
	1	3.7783	6.6059	9.5189	12.3872	15.1643
	2.5	4.3241	7.0599	9.8736	12.6763	15.4104
0.10	0	3.1946	6.1014	8.8440	11.3447	13.6141
	0.5	3.5140	6.3002	8.9911	11.4674	13.7240
	1	3.7644	6.4818	9.1314	11.5863	13.8314
	2.5	4.3122	6.9507	9.5174	11.9225	14.1391
0.20	0	3.1281	5.6843	7.8443	9.6596	11.2236
	0.5	3.4590	5.9174	8.0442	9.8509	11.4159
	1	3.7154	6.1257	8.2301	10.0313	11.5985
	2.5	4.2710	6.6480	8.7217	10.5192	12.0978

Table 8

Non-dimensional frequency parameters for a TB with CC boundary conditions,
($\overline{K}_w = 1$, $\nu = 0.3$, $K = 5/6$, and N=14)

Height-to-length ratio	Foundation parameters	Frequency parameters				
	$\overline{K}_p$	λ_1	λ_2	λ_3	λ_4	λ_5
0.002	0	4.7323	7.8535	10.9952	14.1360	17.2767
	0.5	4.8691	7.9681	11.0858	14.2103	17.3395
	1	4.9945	8.0777	11.1742	14.2835	17.4016
	2.5	5.3191	8.3813	11.4274	14.4965	17.5841
0.05	0	4.6923	7.7041	10.6403	13.4612	16.1591
	0.5	4.8305	7.8222	10.7363	13.5426	16.2307
	1	4.9570	7.9350	10.8296	13.6226	16.3014
	2.5	5.2840	8.2457	11.0957	13.8541	16.4463
0.10	0	4.5821	7.3318	9.8564	12.1455	14.2325
	0.5	4.7250	7.611	9.9687	12.2484	14.3303
	1	4.8548	7.5835	10.0773	12.3487	14.4260
	2.5	5.1875	7.9172	10.3830	12.6358	14.7024
0.20	0	4.2452	6.4188	8.2857	9.9040	11.3489
	0.5	4.4087	6.5930	8.4594	10.0819	11.5325
	1	4.5542	6.7539	8.6230	10.2505	11.7070
	2.5	4.9172	7.1756	9.0633	10.7095	12.1843

Table 9

Non-dimensional frequency parameters for a TB with CC boundary conditions,
($\overline{K}_w = 10$, $\nu = 0.3$, $K = 5/6$, and N=14)

Height-to-lengthratio	Foundation parameters	Frequency parameters				
	$\overline{K}_p$	λ_1	λ_2	λ_3	λ_4	λ_5
0.002	0	4.7534	7.8581	10.9969	14.1368	17.2771
	0.5	4.8885	7.9725	11.0875	14.2111	17.3399
	1	5.0125	8.0820	11.1759	14.2843	17.4020
	2.5	5.3349	8.3851	11.4289	14.4972	17.5845
0.05	0	4.7139	7.7089	10.6422	13.4621	16.1596
	0.5	4.8503	7.8269	10.7381	13.5435	16.2312
	1	4.9753	7.9394	10.8314	13.6234	16.3020
	2.5	5.2991	8.2496	11.0973	13.8550	16.5084
0.10	0	4.6051	7.3374	9.8586	12.1466	14.2332
	0.5	4.7460	7.4664	9.9709	12.2496	14.3310
	1	4.8742	7.5886	10.07942	12.3499	14.4267
	2.5	5.2035	7.9216	10.3850	12.6368	14.7030
0.20	0	4.2738	6.4269	8.2894	9.9061	11.3503
	0.5	4.4343	6.6005	8.4629	10.0839	11.5337
	1	4.5775	6.7608	8.6262	10.2524	11.7083
	2.5	4.9357	7.1814	9.0661	10.7112	12.1854

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Fig. 4 shows the effect of elastic foundation parameters on the natural frequency of an EBB with CC boundary conditions. As can be seen, Winkler elastic foundation parameter has no effect on the natural frequency when it is less than 100, however it has a significant effect for higher values. The effect of shearing parameter of foundation is the same for all values of $\overline{K}_w$.

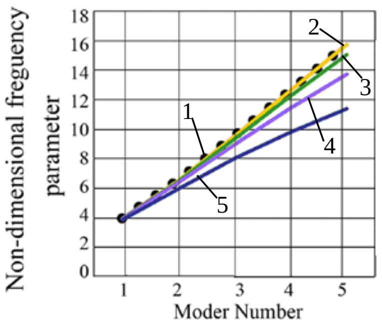

Fig. 3. The effect of height-to-length ratio on natural frequencies (SS boundary conditions, $\overline{K}_w$ =100, $\overline{K}_p$ = 0.5): EBB (1); H/L= 0.02 (2); 0.05 (3); 0.1 (4); 0.2 (5)

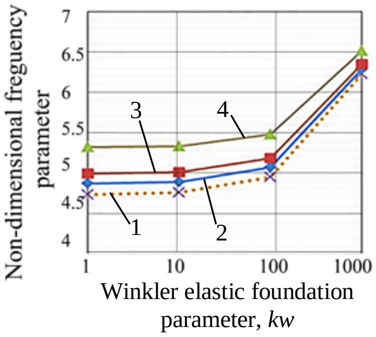

Fig. 4. The effect of elastic foundation parameter on the natural frequency of an EBB (CC boundary conditions): kp= 0 (1); 0.5 (2); 1.0 (3); 2.5 (4)

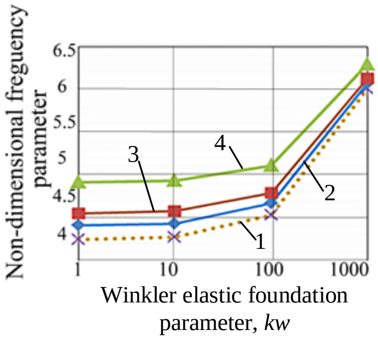

Fig. 5. The effect of elastic foundation parameter on the natural frequency of a TB (CC boundary conditions, H/L=0.2): kp= 0 (1); 0.5 (2); 1.0 (3); 2.5 (4)

Conclusions. In the present paper, the free vibration of EBs/TBs resting on a Pasternak foundation is investigated. Rayleigh-Ritz method is used to obtain the governing equation and the Legendre polynomials multiplied by a boundary function is used as admissible functions. It is shown that the method has a very good and rapid convergence regardless of boundary conditions, beam theory and elastic foundation. Using different theories, the first five eigenvalues of beam on an elastic foundation are tabulated for different Winkler and shearing layer coefficients of foundation and the height-to-length ratios. It is observed that the results of the EBT/TBT for small ratios of the height-to-length are very close. However, as this ratio increases, the results of the EBBT and TBT are deviate depending upon boundary conditions, foundation parameter and the mode number. For a higher mode number, the effect of the height-to-length ratio is more prominent. The effects of elastic foundation parameters have been studied, and it is found that shearing parameter of foundation has no significant effect on the natural frequency of EBBs/TBs.

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